

TAIL ALGEBRAS OF QUANTUM EXCHANGEABLE RANDOM VARIABLES

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ABSTRACT. We show that any countably generated von Neumann algebra with specified normal faithful state can arise as the tail algebra of a quantum exchangeable sequence of noncommutative random variables. We also characterize the cases when the state corresponds to a limit of convex combinations of free products states.

1. INTRODUCTION

Exchangeability is a basic distributional symmetry in probability. It means that the distribution of a sequence of random variables is invariant under finite permutations of these random variables. The de Finetti theorem characterizes an exchangeable infinite sequence of random variables to be identically distributed and conditionally independent over its tail σ -algebra. An alternative formulation of this famous theorem is that the law of such a sequence is a mixture of infinite product measures, where the mixture is specified by a certain random probability measure (see e.g. [4]). Hewitt and Savage [3] cast this theorem in terms of symmetric measures on infinite Cartesian products of a compact Hausdorff space S , and inspired Størmer's transfer [8] of their approach to a C^* -algebraic setting: symmetric states on an infinite tensor product of a C^* -algebra with itself are identified as a mixture of infinite product states. In analogy with the classical situation, here the mixture is specified by a probability measure on the state space of the C^* -algebra.

Recently a noncommutative de Finetti theorem was discovered by Köstler and Speicher [6] in the realm of Voiculescu's free probability theory (see [9]). Replacing random variables by operators and the role of permutations by the natural coaction of Wang's quantum permutations [10], the notion of a quantum exchangeable sequence was introduced in a framework of noncommutative probability spaces. In close analogy to the classical case, a quantum exchangeable infinite sequence is characterized as being identically distributed and 'conditionally free' over its tail algebra. Here 'conditional freeness' means 'freeness with amalgamation'.

In this note we have a closer look at tail algebras of quantum exchangeable sequences. Our main results are:

- ◊ Any countably generated von Neumann algebra may appear as the tail algebra of a quantum exchangeable sequence of selfadjoint operators (see Theorem 3.3).

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- ◊ The tail algebra of a quantum exchangeable sequence lies in the center of the von Neumann algebra generated by the sequence if and only if the corresponding state is a limit of convex combinations of free product states (see Proposition 4.4).
- ◊ There exist quantum exchangeable sequences of projections generating a finite factor and whose tail algebra is nontrivial and abelian (see Example 4.5). Thus, the corresponding state is not a limit of convex combinations of free product states.

Altogether these results show that, as to be expected, the structure of tail algebras of quantum exchangeable infinite sequences has a much higher complexity than those of tail σ -algebras of exchangeable sequences in probability theory.

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2. PRELIMINARIES

A *noncommutative probability space* is a pair (A, ϕ) , where A is a unital algebra over the complex numbers and ϕ is a linear functional on A sending 1 to 1, and the elements of A are called *noncommutative random variables*. A *W^* -noncommutative probability space* is one in which A is a von Neumann algebra and ϕ is a normal state. The *Noncommutative de Finetti Theorem* of Köstler and Speicher [6], states that a sequence $(x_i)_{i=1}^\infty$ of noncommutative random variables in a W^* -noncommutative probability space is quantum exchangeable if and only if the sequence is free over the tail algebra of the sequence, with respect to the ϕ -preserving conditional expectation. Quantum exchangeability of the sequence is defined in terms of a natural coaction of the quantum permutation group of Wang [10]. (See [6] for more on this). The *tail algebra* is the von Neumann algebra $\bigcap_{n=1}^\infty W^*(\{x_i \mid i \geq n\})$ and the ϕ -preserving conditional expectation E from $W^*(\{x_i \mid i \geq 1\})$ onto the tail algebra is guaranteed to exist. This was proved in [5], see also Proposition 4.2 of [6].

The following is the “hard part” of the noncommutative de Finetti theorem (Thm. 1.1 of [6]).

Theorem 2.1 ([6]). *Let $(x_i)_{i=1}^\infty$ be a quantum exchangeable sequence in the W^* -noncommutative probability space (\mathcal{M}, ϕ) , where ϕ is faithful, and suppose \mathcal{M} is generated by $\{x_i \mid i \in I\}$. Let \mathcal{N} be the tail algebra and let $E : \mathcal{M} \rightarrow \mathcal{N}$ be the ϕ -preserving conditional expectation onto \mathcal{N} . Let \mathcal{A}_i be the von Neumann subalgebra of \mathcal{M} generated by $\mathcal{N} \cup \{x_i\}$. Then the family $(\mathcal{A}_i)_{i \in I}$ is free with amalgamation over \mathcal{N} , with respect to E and, therefore,*

$$(\mathcal{M}, E) \cong (*_{\mathcal{N}})_{i=1}^\infty (\mathcal{A}_i, E_i) \quad (1)$$

is isomorphic to the W^ -amalgamated free product of countably infinitely many copies of (\mathcal{A}_1, E_1) , where $E_i : \mathcal{A}_i \rightarrow \mathcal{N}$ is the restriction to \mathcal{A}_i of E .*

The next result is the “easy part” of the noncommutative de Finetti theorem (Prop. 3.1 of [6]):

Proposition 2.2 ([6]). *Let*

$$(\mathcal{M}, E) \cong (*_{\mathcal{B}})_{i=1}^{\infty}(\mathcal{A}_i, E_i)$$

be an amalgamated free product of von Neumann algebras, where every E_i is faithful and normal. Let ϕ be any normal, faithful state on \mathcal{B} and denote also by ϕ the state $\phi \circ E$ on \mathcal{M} . If $x_i \in \mathcal{A}_i$ is such that the moments $E_i(x_i b_1 x_i b_2 \cdots b_{n-1} x_i)$ for all $n \in \mathbf{N}$ and $b_1, \dots, b_{n-1} \in \mathcal{B}$ are independent of i , then the sequence $(x_i)_{i=1}^{\infty}$ in the noncommutative probability space (\mathcal{M}, ϕ) is quantum exchangeable.

In this note, we will prove (Theorem 3.3) that every countably generated von Neumann algebra \mathcal{N} with specified normal faithful state ϕ can arise as the tail algebra of a quantum exchangeable sequence in a W^* -noncommutative probability space $(\mathcal{M}, \phi_{\mathcal{M}})$, in such a way that the restriction of $\phi_{\mathcal{M}}$ to \mathcal{N} is ϕ .

We also investigate the question, given a quantum exchangeable sequence $(x_i)_{i=1}^{\infty}$ in a W^* -noncommutative probability space (\mathcal{M}, ϕ) , of when the distribution of $(x_i)_{i=1}^{\infty}$ under ϕ is a limit of convex combinations of equidistributed free product states. Naturally enough, this occurs if and only if the tail algebra of the sequence commutes with all x_i . See Section 4 for more details. This may be compared to Størmer's result [8], that in the commutative context, all symmetric states are limits of convex combinations of tensor powers.

3. EXAMPLES OF TAIL ALGEBRAS

Now we investigate the content of the tail algebra in the construction of Proposition 2.2.

Proposition 3.1. *In the setting of Proposition 2.2, the tail algebra \mathcal{N} of the sequence $(x_i)_{i=1}^{\infty}$ is a von Neumann subalgebra of \mathcal{B} and is, in fact, the smallest unital von Neumann subalgebra \mathcal{B}_0 of \mathcal{B} satisfying*

$$E(b_0 x^{k_1} b_1 x^{k_2} \cdots b_{n-1} x^{k_n} b_n) \in \mathcal{B}_0. \quad (2)$$

whenever $n \in \mathbf{N}$, $k_1, \dots, k_n \in \mathbf{N}$, $b_0, b_1, \dots, b_n \in \mathcal{B}_0$ and $x = x_i$. (Note that the above expression is independent of the choice of $i \in \mathbf{N}$.)

Proof. If we consider the structure of $L^2(\mathcal{M}, \phi)$, we easily see that the tail algebra \mathcal{N} is a von Neumann subalgebra of \mathcal{B} . Indeed, from Voiculescu's free product construction (see [1] for more detailed discussion in the case of von Neumann algebras),

$$L^2(\mathcal{M}, \phi) = L^2(\mathcal{B}, \phi) \oplus \bigoplus_{\substack{n \geq 1 \\ i_1, \dots, i_n \geq 1 \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_{\phi}} L^2(\mathcal{B}, \phi), \quad (3)$$

where F_i is the Hilbert \mathcal{B} -module $L^2(\mathcal{A}_i, E_i) \ominus \mathcal{B}$ and π_{ϕ} is the Gelfand–Naimark–Segal representation of \mathcal{B} on $L^2(\mathcal{B}, \phi)$. In particular, the image in $L^2(\mathcal{M}, \phi)$ of the von Neumann algebra generated by $\{x_i \mid i \geq N\}$ is contained in

$$L^2(\mathcal{B}, \phi) \oplus \bigoplus_{\substack{n \geq 1 \\ i_1, \dots, i_n \geq N \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_{\phi}} L^2(\mathcal{B}, \phi)$$

and the intersection of these spaces is $L^2(\mathcal{B}, \phi)$. We must, therefore, have $\mathcal{N} \subseteq \mathcal{B}$.

Since $\mathcal{N} \subseteq \mathcal{B} \cap W^*(\{x_i \mid i \geq 1\})$, we have

$$\mathcal{N} \subseteq E(W^*(\{x_i \mid i \geq 1\})).$$

Thus, we have

$$\mathcal{N} \subseteq W^*(\{E(x_{i_1}x_{i_2} \cdots x_{i_n}) \mid n \geq 1, i_1, \dots, i_n \geq 1\}). \quad (4)$$

To see that we have equality in (4), take $j_1, \dots, j_N \geq 1$ and let $M_1 = \min(j_1, \dots, j_N)$ and $M_2 = \max(j_1, \dots, j_N)$. Then $x_{j_1}x_{j_2} \cdots x_{j_N} = b + y$, where $b = E(x_{j_1}x_{j_2} \cdots x_{j_N})$ and where the element \hat{y} of $L^2(\mathcal{M}, \phi)$ corresponding to y belongs to

$$\bigoplus_{\substack{1 \leq n \leq N \\ M_1 \leq i_1, \dots, i_n \leq M_2 \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi).$$

By exchangeability, for each $p \geq 0$ we have $E(x_{j_1+p}x_{j_2+p} \cdots x_{j_N+p}) = E(x_{j_1}x_{j_2} \cdots x_{j_N})$ and, therefore, $x_{j_1+p}x_{j_2+p} \cdots x_{j_N+p} = b + y_p$ where

$$\hat{y}_p \in \bigoplus_{\substack{1 \leq n \leq N \\ M_1+p \leq i_1, \dots, i_n \leq M_2+p \\ i_j \neq i_{j+1}}} F_{i_1} \otimes_{\mathcal{B}} F_{i_2} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} F_{i_n} \otimes_{\pi_\phi} L^2(\mathcal{B}, \phi). \quad (5)$$

Now since the subspaces (5) corresponding to values of p that differ by more than $M_2 - M_1$ are orthogonal to each other, we see that for each $q \in \mathbf{N}$, the ergodic averages

$$\frac{1}{K} \sum_{p=q}^{q+K-1} x_{i_1+p}x_{i_2+p} \cdots x_{i_N+p}$$

converge in $L^2(\mathcal{M}, \phi)$ to b as $K \rightarrow \infty$. So $b \in W^*(\{x_i \mid i \geq q\})$ for all $q \geq 1$. Letting $q \rightarrow \infty$, we get $b = E(x_{j_1}x_{j_2} \cdots x_{j_N}) \in \mathcal{N}$. This proves

$$\mathcal{N} = W^*(\{E(x_{i_1}x_{i_2} \cdots x_{i_n}) \mid n \geq 1, i_1, \dots, i_n \geq 1\}). \quad (6)$$

It remains to show $\mathcal{N} = \mathcal{B}_0$. Using (6), we have $E(b_0x^{k_1}b_1x^{k_2} \cdots b_{n-1}x^{k_n}b_n) \in \mathcal{N}$ whenever $n \in \mathbf{N}$, $k_1, \dots, k_n \in \mathbf{N}$, $b_0, b_1, \dots, b_n \in \mathcal{N}$ and $x = x_i$. By the characterization of \mathcal{B}_0 , this implies $\mathcal{B}_0 \subseteq \mathcal{N}$. So it suffices to see that we have $E(x_{i_1}x_{i_2} \cdots x_{i_n}) \in \mathcal{B}_0$ for all $n \geq 1$ and $i_1, \dots, i_n \geq 1$. However, this follows by considering how mixed moments of the free variables x_1, x_2, \dots can be evaluated in terms the moments of the individual x_i . Speicher's operator-valued cumulants [7] can be used to give a careful proof of this fact. Indeed, using the expression of the cumulants in terms of moments and the Möbius function and using the defining property (2) of \mathcal{B}_0 , we see that for each individual $x = x_i$, the operator valued cumulant $\kappa[xb_1, xb_2, \dots, xb_{n-1}, x]$ belongs to \mathcal{B}_0 for every $b_1, \dots, b_{n-1} \in \mathcal{B}_0$. Now using the moment cumulant formula for $E(x_{i_1}x_{i_2} \cdots x_{i_n})$ and the vanishing of mixed cumulants, we obtain $E(x_{i_1}x_{i_2} \cdots x_{i_n}) \in \mathcal{B}_0$. \square

Lemma 3.2. *Let \mathcal{N} be a countably generated von Neumann algebra equipped with a normal faithful state ϕ . Then there is a von Neumann algebra \mathcal{A} containing \mathcal{N} as a unital von Neumann subalgebra and possessing a ϕ -preserving conditional expectation*

$E : \mathcal{A} \rightarrow \mathcal{N}$ onto \mathcal{N} , and there is a self-adjoint element $a \in \mathcal{A}$ with the property that $\{E(a^k) \mid k \in \mathbf{N}\}$ generates \mathcal{N} as a von Neumann algebra.

Proof. One easily sees (using the spectral theorem) that the von Neumann algebra \mathcal{N} is also generated by a finite or countable collection $(p_i)_{i \in I}$ of projections. Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{N}$, with $\mathcal{N} \subseteq \mathcal{A}$ identified with the constant sequences and with the conditional expectation E given by some strictly positive weights $(\alpha_i)_{i \in I}$ that sum to 1:

$$E((x_i)_{i \in I}) = \sum_i \alpha_i x_i.$$

We let $a = (\beta_i p_i)_{i \in I} \in \mathcal{A}$ for some bounded family $\beta_i \in \mathbf{R}$; thus, we have

$$E(a^k) = \sum_i \alpha_i \beta_i^k p_i.$$

If I is finite, then by choosing all the β_i to be distinct, the determinant of the matrix $(\beta_i^j)_{i \in I, 0 \leq j < |I|}$, being a Vandermonde determinant, is seen to be nonzero. Thus, we recover $\{p_i \mid i \in I\}$ by taking linear combinations of $(E(a^k))_{0 \leq k < |I|}$.

Suppose I is infinite, identify it with \mathbf{N}_0 and let $\beta_i = 2^{-i}$. Let $\mathcal{D} = W^*(\{E(a^k) \mid k \in \mathbf{N}\})$. Then

$$\lim_{k \rightarrow \infty} E(a^k) = \lim_{k \rightarrow \infty} \left(\alpha_0 p_0 + \sum_{i=1}^{\infty} \alpha_i 2^{-ki} p_i \right) = \alpha_0 p_0,$$

where the convergence is in norm topology, and $p_0 \in \mathcal{D}$. Similarly,

$$\begin{aligned} \lim_{k \rightarrow \infty} 2^k (E(a^k) - \alpha_0 p_0) &= \alpha_1 p_1, \\ \lim_{k \rightarrow \infty} 2^{2k} (E(a^k) - \alpha_0 p_0 - 2^{-1} \alpha_1 p_1) &= \alpha_2 p_2 \end{aligned}$$

and so on. Thus, we get $p_0, p_1, p_2, \dots \in \mathcal{D}$ and \mathcal{D} is all of \mathcal{N} . \square

Theorem 3.3. *Let \mathcal{N} be a countably generated von Neumann algebra with a normal, faithful state ϕ . Then there is a quantum exchangeable sequence x_1, x_2, \dots in some W^* -noncommutative probability space $(\mathcal{M}, \phi_{\mathcal{M}})$ whose tail algebra is a copy of \mathcal{N} in \mathcal{M} , with the restriction of $\phi_{\mathcal{M}}$ to \mathcal{N} being ϕ .*

Proof. Let $\mathcal{A} \supseteq \mathcal{N}$ and $a \in \mathcal{A}$ be as from Lemma 3.2. For each $j \in \mathbf{N}$, let \mathcal{A}_j be a copy of \mathcal{A} with the ϕ -preserving conditional expectation onto \mathcal{N} denoted by E_j . Let

$$(\mathcal{M}, E) = (*_{\mathcal{N}})_{j=1}^{\infty} (\mathcal{A}_j, E_j)$$

be their amalgamated free product of von Neumann algebra and, of course, let $\phi_{\mathcal{M}} = \phi \circ E$. Let x_j be the copy of a in the j th copy \mathcal{A}_j of \mathcal{A} . By Proposition 2.2, $(x_j)_{j=1}^{\infty}$ is a quantum exchangeable sequence. By Lemma 3.2, the set

$$\{E(x_j^k) \mid k \in \mathbf{N}\} \tag{7}$$

generates all of \mathcal{N} . By Proposition 3.1, the tail algebra of the sequence $(x_i)_{i \in I}$ is equal to \mathcal{N} . \square

4. FREE PRODUCT STATES

Given a C^* -algebra A with state ϕ and with self-adjoint elements $x_i \in A$, ($i \in I$), we wish to regard the variables x_i abstractly, independently of their realization in A . Consider the $*$ -algebra $\mathbf{C}\langle X \rangle = \mathbf{C}\langle X_i \mid i \in I \rangle$ of polynomials in noncommuting variables X_i , with the involution defined so that $X_i^* = X_i$, and consider the unital $*$ -algebra representation $\alpha : \mathbf{C}\langle X \rangle \rightarrow A$ that sends X_i to x_i .

Definition 4.1. By a *free product state* on $\mathbf{C}\langle X \rangle$, we will mean a functional ψ of $\mathbf{C}\langle X \rangle$ such that $\psi(1) = 1$ and the variables $(X_i)_{i \in I}$ are free with respect to ψ . If, in addition, for all k the moments $\psi(X_i^k)$ are independent of i , then we say ψ is an *equidistributed free product state* on $\mathbf{C}\langle X \rangle$. We will say that a linear functional ϕ of $\mathbf{C}\langle X \rangle$ is a *limit of convex combinations of uniformly bounded free product states* (respectively, of uniformly bounded equidistributed free product states) if for some choice of constants $C_i > 0$, the functional ϕ is the limit in the topology of pointwise convergence on $\mathbf{C}\langle X \rangle$ of convex combinations of free product states (respectively, of equidistributed free product states) ψ on $\mathbf{C}\langle X \rangle$ that satisfy $|\psi(X_i^k)| \leq C_i^k$ for all $i \in I$ and $k \in \mathbf{N}$.

The following result shows that a state that is a limit of convex combinations of uniformly bounded free product states arises from the situation of a free product with amalgamation over a subalgebra of the center.

Proposition 4.2. *Let A be a C^* -algebra with state ϕ . Suppose $x_i \in A$ ($i \in I$), for I countable, are self-adjoint elements that together generate A as a C^* -algebra and so that $\tilde{\phi} := \phi \circ \alpha$ is a limit of convex combinations of uniformly bounded free product states with respect to $(x_i)_{i \in I}$, where $\alpha : \mathbf{C}\langle X \rangle \rightarrow A$ is the $*$ -homomorphism given by $X_i \mapsto x_i$. Then there is a unital C^* -algebra B with a unital C^* -subalgebra $D \subseteq Z(B)$ of the center of B and with a faithful conditional expectation $E : B \rightarrow D$, and there is a unital $*$ -homomorphism $\pi : A \rightarrow B$ and a faithful tracial state $\sigma = \sigma \circ E$ on B so that $\phi = \sigma \circ \pi$ and the family $(\pi(x_i))_{i \in I}$ is free with respect to E .*

Furthermore, if $\tilde{\phi}$ is a limit of convex combinations of uniformly bounded equidistributed free product states with respect to $(x_i)_{i \in I}$, then $E : B \rightarrow D$ and $\pi : A \rightarrow B$ may be chosen so that for all n and all $d_0, \dots, d_n \in D$ the moment $E(d_0 \pi(x_i) d_1 \cdots \pi(x_i) d_n)$ of the variable $\pi(x_i)$ is independent of i .

Proof. We fix constants C_i as in Definition 4.1. If ψ is a free product state on $\mathbf{C}\langle X \rangle$, then the usual Gelfand–Naimark–Segal construction yields a $*$ -representation $\pi_\psi : \mathbf{C}\langle X \rangle \rightarrow B(\mathcal{H}_\psi)$ with $\|\pi_\psi(X_i)\| \leq C_i$. Let $B_\psi \subseteq B(\mathcal{H}_\psi)$ be the unital C^* -algebra generated by the image of π_ψ and denote also by ψ the state on B_ψ so that $\psi \circ \pi_\psi$ is the original functional ψ on $\mathbf{C}\langle X \rangle$. Of course, the condition that ψ is a free product state on $\mathbf{C}\langle X \rangle$ implies that the variables $(\pi_\psi(X_i))_{i \in I}$ are free in B_ψ with respect to ψ . Moreover, B_ψ is isomorphic to the free product over the scalars of the abelian C^* -algebras generated by the $\pi_\psi(X_i)$. Since the restrictions of ψ to these abelian C^* -algebras are faithful, and since the free product of faithful states is faithful [2] and since the free product of traces is a trace (see [9]), it follows that ψ is a faithful trace on B_ψ .

By hypothesis, we may write $\tilde{\phi}$ as the limit of a net of convex combinations of free product states, and consider the set F of all the free product states appearing with nonzero coefficients in these convex combinations. Let

$$\tilde{B} = \prod_{\psi \in F} B_\psi = \{(b_\psi) \mid b_\psi \in B_\psi, \sup_{\psi} \|b_\psi\| < \infty\}$$

be the direct product of C^* -algebras. Then $\tilde{\beta} : \mathbf{C}\langle X \rangle \rightarrow \tilde{B}$ defined by $\tilde{\beta}(p) = (\pi_\psi(p))$ is a $*$ -representation. Let $\tilde{D} \subseteq \tilde{B}$ be the subalgebra consisting of all sequences of scalars, i.e. $(\lambda_\psi 1)_{\psi \in F}$ for $\lambda_\psi \in \mathbf{C}$. Clearly, $\tilde{D} \cong \ell^\infty(F)$ is a subalgebra of the center of \tilde{B} , and the map $\tilde{E} : \tilde{B} \rightarrow \tilde{D}$ given by $\tilde{E}((b_\psi)_{\psi \in F}) = (\psi(b_\psi))_{\psi \in F}$ is a conditional expectation that is faithful because each tracial state ψ is faithful on B_ψ . The variables $(\tilde{\beta}(X_i))_{i \in I}$ are free with respect to \tilde{E} .

Note that in the above construction, if all free product states $\psi \in F$ are equidistributed, then the moments

$$\tilde{E}(d_0 \tilde{\beta}(X_i) d_1 \cdots \tilde{\beta}(X_i) d_n) \quad (8)$$

of the variables $\tilde{\beta}(X_i)$ are independent of i .

To a convex combination $\rho = \sum t_\psi \psi$ (with finite support) of elements of F , consider the state $\hat{\rho}$ of \tilde{D} given by the corresponding weighted average, $\hat{\rho} : (\lambda_\psi 1)_{\psi \in F} \mapsto \sum t_\psi \lambda_\psi$. Let $(\rho_j)_{j \in J}$ denote a net of convex combinations of free product states on $\mathbf{C}\langle X \rangle$ that converges (in the topology of pointwise convergence on $\mathbf{C}\langle X \rangle$) to $\tilde{\phi}$. Replacing this net by a subnet, if necessary, we may without loss of generality assume that the corresponding net $\hat{\rho}_j$ converges in the weak*-topology on \tilde{D}^* to a state $\tilde{\sigma}$ on \tilde{D} . Then we have $\tilde{\phi} = \tilde{\sigma} \circ \tilde{E} \circ \tilde{\beta}$.

We are almost done, except that $\tilde{\sigma} \circ \tilde{E}$ need not be faithful. Let $\pi_{\tilde{\sigma} \circ \tilde{E}}$ denote the Gelfand–Naimark–Segal representation associated to the tracial state $\tilde{\sigma} \circ \tilde{E}$ on \tilde{B} , and let $B = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{B})$ and $D = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{D})$. Then the corresponding state σ on B is a faithful trace. We claim that

$$E(\pi_{\tilde{\sigma} \circ \tilde{E}}(x)) = \pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{E}(x))$$

defines a faithful conditional expectation (i.e., projection of norm 1), $E : B \rightarrow D$. To verify this, let us identify \tilde{D} with $C(\Omega)$, where Ω is the spectrum of \tilde{D} . Then $\tilde{\sigma}$ is given by integration against a Borel probability measure ν on Ω . Given $x = (x_\psi)_{\psi \in F} \in \tilde{B}$, let $N(x)$ denote the element $(\|x_\psi\|)_{\psi \in F}$ of \tilde{D} . We easily verify that $\|\pi_{\tilde{\sigma} \circ \tilde{E}}(x)\|$ equals the essential supremum of $N(x)$ taken with respect to the measure ν . On the other hand, the norm of $\|\pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{E}(x))\|$ equals the essential supremum of $|\tilde{E}(x)|$ taken with respect to the measure ν . Since $|\tilde{E}(x)| \leq N(x)$, it is clear that E is a projection of norm 1. Moreover, we easily check $\sigma \circ E = \sigma$ so faithfulness of σ implies that E is faithful.

Let $\beta = \pi_{\tilde{\sigma} \circ \tilde{E}} \circ \tilde{\beta} : \mathbf{C}\langle X \rangle \rightarrow B$. Then $\tilde{\phi} = \phi \circ \alpha = \sigma \circ \beta$. We will now define a $*$ -homomorphism $\pi : A \rightarrow B$ so that all triangles in the diagram

$$\begin{array}{ccc} \mathbf{C}\langle X \rangle & \xrightarrow{\alpha} & A \\ \beta \downarrow & \nearrow \pi & \downarrow \phi \\ B & \xrightarrow{\sigma} & \mathbf{C} \end{array}$$

commute. Indeed, since ϕ and σ are faithful, we have

$$\begin{aligned} \|a\| &= \limsup_{n \rightarrow \infty} |\phi((a^*a)^n)|^{1/2n}, & (a \in A) \\ \|b\| &= \limsup_{n \rightarrow \infty} |\sigma((b^*b)^n)|^{1/2n}, & (b \in B). \end{aligned}$$

But this implies $\|\alpha(p)\| = \|\beta(p)\|$ for all $p \in \mathbf{C}\langle X \rangle$. So the $*$ -homomorphism defined on the image of α by $\alpha(p) \mapsto \beta(p)$ is isometric and extends to an isometric $*$ -homomorphism $\pi : A \rightarrow B$, as required.

It remains to show that the family $(\beta(X_i))_{i \in I}$ is free with respect to E . As is well known, there is a list $(Q_k)_{k \geq 1}$ of universal polynomials (Q_k in $n(k)^2$ variables) such that elements $(a_i)_{i=1}^\infty$ of an algebra are free with respect to a linear functional ψ if and only if

$$Q_k(\psi(a_1), \dots, \psi(a_1^{n(k)}), \psi(a_2), \dots, \psi(a_2^{n(k)}), \dots, \psi(a_{n(k)}), \dots, \psi(a_{n(k)}^{n(k)})) = 0$$

for all k . It is apparent that, in \tilde{D} , we have

$$\begin{aligned} Q_k(\tilde{E} \circ \tilde{\beta}(X_1), \dots, \tilde{E} \circ \tilde{\beta}(X_1^{n(k)}), \tilde{E} \circ \tilde{\beta}(X_2), \dots, \tilde{E} \circ \tilde{\beta}(X_2^{n(k)}), \dots, \\ \tilde{E} \circ \tilde{\beta}(X_{n(k)}), \dots, \tilde{E} \circ \tilde{\beta}(X_{n(k)}^{n(k)})) = 0 \end{aligned}$$

for all k . Applying the $*$ -homomorphism $\pi_{\tilde{\sigma} \circ \tilde{E}}$ to this and using

$$\pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{E} \circ \tilde{\beta}(X_i^k)) = E(\pi_{\tilde{\sigma} \circ \tilde{E}}(\tilde{\beta}(X_i^k))) = E \circ \beta(X_i^k),$$

we get

$$\begin{aligned} Q_k(E \circ \beta(X_1), \dots, E \circ \beta(X_1^{n(k)}), E \circ \beta(X_2), \dots, E \circ \beta(X_2^{n(k)}), \dots, \\ E \circ \beta(X_{n(k)}), \dots, E \circ \beta(X_{n(k)}^{n(k)})) = 0 \end{aligned}$$

for all k .

In the case that all free product states are equidistributed, the observation above regarding the moments (8) of $\tilde{\beta}(X_i)$ with respect to \tilde{E} implies that the moments of $\pi(x_i)$ with respect to E are independent of i . \square

Lemma 4.3. *Let $D = C(\Omega)$ be a commutative, unital C^* -algebra and for every $i \in I$ let A_i be a unital C^* -algebra with D embedded as a unital C^* -subalgebra of the center, $Z(A_i)$, of A_i . Assume $E_i : A_i \rightarrow D$ is a conditional expectation whose Gelfand–Naimark–Segal representation is faithful. Let*

$$(A, E) = (*_D)_{i \in I}(A_i, E_i)$$

be a reduced amalgamated free product of C^* -algebras. Take $\omega \in \Omega$, consider the character $\text{ev}_\omega : f \mapsto f(\omega)$ of D and let $\rho_\omega = \text{ev}_\omega \circ E$. Then the family $(A_i)_{i \in I}$ is free with respect to ρ_ω .

Proof. Suppose $a_j \in A_{i_j} \cap \ker \rho_\omega$ for $j = 1, \dots, n$ and $i_j \neq i_{j+1}$ for all $1 \leq j < n$. We must show $\rho_\omega(a_1 a_2 \cdots a_n) = 0$. Let $\epsilon > 0$. Since $E(a_j) \in C(\Omega)$ vanishes at ω , by Tietze's extension theorem there is $f \in C(\Omega)$ such that $f(\omega) = 1$ and $\|E(a_j)f\| < \epsilon$ for all j . Let $f_j = E(a_j)f$. Then $E((fa_1 - f_1)(fa_2 - f_2) \cdots (fa_n - f_n)) = 0$, so

$$\begin{aligned} \rho_\omega((a_1 - f_1) \cdots (a_n - f_n)) &= f \rho_\omega((a_1 - f_1) \cdots (a_n - f_n)) \\ &= \rho_\omega((fa_1 - f_1) \cdots (fa_n - f_n)) = 0. \end{aligned}$$

But if $M = 1 + \epsilon + \max_j \|a_j\|$, then

$$|\rho_\omega(a_1 a_2 \cdots a_n) - \rho_\omega((a_1 - f_1) \cdots (a_n - f_n))| \leq nM^{n-1}\epsilon.$$

Letting $\epsilon \rightarrow 0$ finishes the proof. \square

Proposition 4.4. *Let $(x_i)_{i \in I}$ be quantum exchangeable random variables in (\mathcal{M}, ϕ) , suppose \mathcal{M} is generated by $\{x_i \mid i \in I\}$ and let $\alpha : \mathbf{C}\langle X \rangle \rightarrow \mathcal{M}$ be the $*$ -homomorphism given by $\alpha(X_i) = x_i$. Then $\phi \circ \alpha$ is a limit of convex combinations of uniformly bounded equidistributed free product states with respect to $(x_i)_{i \in I}$ if and only if the tail algebra \mathcal{N} lies in the center of \mathcal{M} .*

Proof. Suppose the tail algebra lies in the center of \mathcal{M} . Then by Theorem 2.1, \mathcal{M} is the free product with amalgamation over the tail algebra \mathcal{N} , and $\phi = \phi|_{\mathcal{N}} \circ E$, where $E : \mathcal{M} \rightarrow \mathcal{N}$ is the ϕ -preserving conditional expectation with respect to which the algebras $\mathcal{A}_i = W^*(\mathcal{N} \cup \{x_i\})$ are free over \mathcal{N} . Regarding \mathcal{N} as a commutative C^* -algebra, by the Gelfand theorem, we have $\mathcal{N} \cong C(\Omega)$. Then, by a classical result, every state on $C(\Omega)$ lies in the closed convex hull of the set of point evaluation maps $\{\text{ev}_\omega \mid \omega \in \Omega\}$. By Lemma 4.3, every functional $\text{ev}_\omega \circ E \circ \alpha$ is an equidistributed free product state on $\mathbf{C}\langle X \rangle$ and clearly the boundedness criterion is satisfied with constants $C_i = \|x_i\|$. Taking convex combinations of the ev_ω that approximate $\phi|_{\mathcal{N}}$, we easily see that $\phi \circ \alpha$ is a limit of convex combinations of equidistributed free product states.

Conversely, suppose that $\phi \circ \alpha$ is a limit of convex combinations of uniformly bounded equidistributed free product states. Consider the C^* -subalgebra $C^*(\{x_i \mid i \in I\})$ of \mathcal{M} generated by the x_i . By Proposition 4.2, there is a unital C^* -algebra B and a unital subalgebra D of the center of B with a faithful conditional expectation $E : B \rightarrow D$ and a unital, injective $*$ -homomorphism $\pi : C^*(\{x_i \mid i \in I\}) \rightarrow B$ so that the elements $(\pi(x_i))_{i \in I}$ are free with respect to E , and so that the moments of $\pi(x_i)$ are independent of i ; furthermore, there is a faithful state σ on D so that $\sigma \circ E \circ \pi = \phi|_{\mathcal{A}}$. Let \mathfrak{A} be the C^* -subalgebra of B generated by $D \cup \{\pi(x_i) \mid i \in I\}$. Then \mathfrak{A} is an amalgamated free product of C^* -algebras $C^*(D \cup \{x_i\})$, with amalgamation over D . We take the Hilbert space representation ρ of \mathfrak{A} that is the Gelfand–Naimark–Segal construction for the restriction of the state $\sigma \circ E$ to \mathfrak{A} . The strong-operator-topology closure of $\rho(\mathfrak{A})$ is a von Neumann algebra, \mathcal{Q} , that is isomorphic to a free product

$$(*_{\mathcal{D}})_{i \in I}(W^*(\mathcal{D} \cup \{\rho(x_i)\}, \overline{E}) \quad (9)$$

with amalgamation over the strong-operator-topology closure \mathcal{D} of $\rho(D)$. Taking strong-operator-topology limits (using Kaplansky's density theorem), we easily see that \mathcal{D} lies in the center of \mathcal{Q} . Since $\sigma \circ E$ and ϕ are faithful, the composition $\rho \circ \pi$, when compressed to a Hilbert subspace, equals the Gelfand–Naimark–Segal construction of the restriction of ϕ to \mathfrak{A} . Therefore, this mapping of \mathfrak{A} into the strong-operator-closure of \mathfrak{A} is canonically isomorphic to the inclusion of \mathfrak{A} in \mathcal{M} , and we may regard \mathcal{M} as embedded in the amalgamated free product (9). By Proposition 3.1, the tail algebra of $\{x_i \mid i \in I\}$ lies in \mathcal{D} . Thus, the tail algebra is in the center of \mathcal{M} . \square

The following easy example shows that the tail algebra can be commutative without lying in the center of the algebra generated by the quantum exchangeable sequence.

Example 4.5. Let $\mathcal{B} = \mathbf{C} \oplus \mathbf{C}$ embed into $M_2(\mathbf{C})$ as the diagonal matrices. For each $i \in \mathbf{N}$, let \mathcal{A}_i be a copy of $M_2(\mathbf{C})$ and let $E_i : \mathcal{A}_i \rightarrow \mathcal{B}$ be the conditional expectation taking a matrix to its diagonal. Let

$$(\mathcal{M}, E) = (*_{\mathcal{B}})_{i=1}^{\infty}(\mathcal{A}_i, E_i)$$

be the amalgamated free product of von Neumann algebras. We note that \mathcal{M} is easily seen to be isomorphic to the free group factor $L(\mathbf{F}_{\infty})$. Let ψ be any faithful state on \mathcal{B} . Then $\phi = \psi \circ E$ is a normal faithful state on \mathcal{M} . Fix $0 < t < 1/2$ and let x_i be the copy of the projection $\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$ in \mathcal{A}_i . By Proposition 2.2, the sequence $(x_i)_{i=1}^{\infty}$ is quantum exchangeable. Applying Proposition 3.1, we see that the tail algebra \mathcal{N} of the sequence is $\mathcal{B} \cong \mathbf{C} \oplus \mathbf{C}$ and, incidentally, the von Neumann algebra generated by the sequence $\{x_i \mid i \in \mathbf{N}\}$ is all of \mathcal{M} .

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